UNIVERSITY OF THE PUNJAB

Part-I **A/2016**

Examination:- M.A./M.Sc.

Roll No. ....

Subject: Mathematics (Old & New Course) **TIME ALLOWED: 3 hrs.**

PAPER: I (Real Analysis) **MAX. MARKS: 100**

Note: Attempt FIVE questions in all, selecting at least TWO from each Section.

SECTION- I

Q. 1 **(a)** Let $b < 0$, and $bS = \{bs | s \in S\}$ then prove that $\inf(bS) = b\sup(S)$.

(b) State and prove Archimedean property of real numbers.

(c) prove that set of natural numbers is unbounded above.**(8+6+6)**

Q.2 **(a)** State and prove Cauchy condensation test for the convergence of series. Apply it to test the convergence of $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$.

(b) Prove that every bounded sequence of real numbers has a convergent subsequence.

(c) Prove that a monotonic bounded sequence of real numbers is convergent. **(10+5+5)**

Q.3 **(a)** For $f(x) = \begin{cases} x+2\,; & -3 < x < -2 \\ -x-2; & -2 \le x < 0 \\ x+2; & 0 \le x < 1 \end{cases}$, Discuss the continuity and discontinuity Of 1st and 2nd kind of $f(x)$.

(b) Suppose $f: A \to R$ is uniformly continuous. If $\{x_n\}$ is a Cauchy sequence in A. Prove that $\{f(x_n)\}$is a Cauchy sequence in R.

(c) If $f$and g are continuous at $x = a$ then show that the function $Max\{f, g\}$ and $Min\{f, g\}$ Are continuous at $x = a$.**(8+7+5)**

Q.4 **(a)**State and prove Cauchy Mean Value theorem.

(b) Let f be a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$ then there Exists a point $x \in (a, b)$ such that $f'(x) = \lambda$.

(c) Let $f(x) = \begin{cases} x^3 \sin(\frac{1}{x}) \; ; & x \neq 0 \\ 0 \quad ; & x = 0 \end{cases}$ prove that

(i) $f(x)$ is continuous at $x = 0$.

(ii)$f'(x)$ is continuous at $x = 0$.**(6+6+8)**

Q. 5 **(a)** If f_x and $f_y$exists in some neighborhood of (a, b) and both are differentiable at (a, b) Then prove that$f_{xy}(a, b) = f_{yx}(a, b)$.

(b) Show that the function $f: R^2 \to R$given by $f(x, y) = \begin{cases} \frac{(x+y)\sin(x^2+y^2)}{x^2+y^2}\,; & (x,y) \neq (0,0) \\ 0 \quad ; & (x,y) = (0,0) \end{cases}$

Is continuous at (0,0).**(10+10)**

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SECTION- II

Q.6 (a) Prove that $f(x) = \sin x$ is Riemann integrable over $\left[0, \frac{\pi}{2}\right]$.

(b) If $f \epsilon \Re(\alpha)$ on $I = [a, b]$ and C be any constant then $Cf \epsilon \Re(\alpha)$ on I and

$$\int_a^b Cf\, d\alpha = C \int_a^b f\, d\alpha$$ **(10+10)**

Q.7 (a) Let f be a function of bounded variation on $I = [a, b]$ if f is continuous at c in I, then prove that the function defined by

$$g(x) = \begin{cases} V_a^x; & x \neq a, a < x \leq b \\ 0; & x = a \end{cases}$$

is continuous at c.

(b) If $f \epsilon \Re(\alpha)$ on $I = [a, b]$, then show that $|f| \epsilon \Re(\alpha)$ on I and $\left| \int_a^b f\, d\alpha \right| \leq \int_a^b |f|\, d\alpha$. **(10+10)**

Q. 8 (a) Prove that a series of functions $\sum f_n$ will converge uniformly (and absolutely) on $[a, b]$ if there exists a convergent series $\sum_1^\infty M_n$ of positive numbers such that for all $x \in [a, b]$

$|f_n(x)| \leq M_n$ for all n.

(b) Show that the sequence $\{f_n\}$ where $f_n(x) = \frac{x}{1+nx^2}$ converges uniformly to a function f On $[0,1]$ and the identity $f'(x) = \lim_{n\to\infty} f_n'(x)$ holds for $x \neq 0$ and does not hold for $x = 0$, Explain why this is so?.

(c) Examine $f_n(x) = x^{n-1}(1 - x)$ for uniform convergence.**(7+7+6)**

Q. 9 (a) Prove that every absolutely convergent integral is convergent.

(b) Examine the convergence of given improper integrals

(i) $\int_0^\infty \frac{x^3 dx}{\sqrt{x^8+1}}$ (ii) $\int_1^\infty \frac{\sin kx dx}{x}$ (iii) $\int_0^\infty \frac{\cos x dx}{\sqrt{x^2+x}}$ **(5+5+5+5)**

UNIVERSITY OF THE PUNJAB

Part-I A/2016
Examination:- M.A./M.Sc.

Roll No. ....

Subject: Mathematics (Old & New Course) **TIME ALLOWED: 3 hrs.**
PAPER: II (Algebra) **MAX. MARKS: 100**

NOTE: *Attempt any FIVE questions in all selecting at least two questions from each section.*

		SECTION-I	Marks
Q.1	(a)	Define involution and prove that every group of even order has at least one involution.	(10)
	(b)	Define group homomorphism and prove that homomorphic image of a group and kernel of homomorphism are itself groups.	(10)
Q.2	(a)	Show that any two cyclic groups of the same order are isomorphic.	(10)
	(b)	Prove that the centralizer $C_G(X)$ of a subset X of a group G is a subgroup of G. Find the centre of Dihedral group of order 8.	(10)
Q.3	(a)	The number of elements in a conjugacy class C_a of an element a in a group G is equal to the index of its normalizer in G. Verify the result with example.	(10)
	(b)	Let H, K be finite subgroups of a group G. Then each double coset HaK contains mn/q elements where $m, n,$ and q are the orders of the subgroups $H, K,$ and $Q = H \cap aKa^{-1}$ respectively.	(10)
Q.4	(a)	Let H be a normal subgroup and K a subgroup of a group G. Then HK is a subgroup of G, $H \cap K$ is normal in K and $HK/H \cong K/H \cap K$.	(10)
	(b)	Let G be group and $I(G)$ the group of its inner automorphisms. Then show that $I(G)$ .is normal subgroup of the group of all automorhisms of G.	(10)
Q.5	(a)	If A is a finite group and p a prime divisor of the order of A then prove that A contains an element of order p.	(10)
	(b)	Show that A_n is generated by 3-cycles $(1\ 2\ 3), (1\ 2\ 4), \ldots, (1\ 2\ n)$.	(10)
		SECTION-II	
Q.6	(a)	Distinguish between Division Ring and Field. Explain with examples in detail.	(10)
	(b)	Define Integral domain and prime Ideal. Let R be a commutative ring. Then prove that M is a prime ideal of a ring R if and only if R/M is an integral domain.	(10)
Q.7	(a)	Prove that the non-zero eigenvectors of a matrix A corresponding to distinct eigenvalues are linearly independent.	(10)
	(b)	Prove that a square matrix A is orthogonal if and only if the columns of A form an orthonormal set.	(10)
Q.8	(a)	Prove that every integral domain can be embedded in a field.	(10)
	(b)	State and prove rank-nullity theorem for vector spaces.	(10)
Q.9	(a)	Define eigenvalue and symmetric matrix. Prove that the Eigen values of a symmetric matrix are all real.	(10)
	(b)	Suppose that P is the change of basis matrix from a basis $\{u_i\}$ to a basis $\{w_i\}$ and suppose Q is the change of basis matrix from a basis $\{w_i\}$ to a basis $\{u_i\}$. Prove that P is invertible and that $Q = P^{-1}$.	(10)

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Part-I A/2016
Examination:- M.A./M.Sc.

Roll No. ....

Subject: Mathematics (Old & New Course) **TIME ALLOWED: 3 hrs.**
PAPER: III (Complex Analysis and Differential Geometry) **MAX. MARKS: 100**

NOTE: Attempt any FIVE questions by selecting at least TWO questions from each section.

SECTION-I

Q.1. (a) Prove that the necessary and sufficient conditions for a function $w = f(z)$ to be an analytic function is that the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exist, are continuous and satisfy the Cauchy Riemann equations. Also, prove that $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

(b) Prove that harmonic conjugate of a function is not unique. Also prove that if a function is analytic in a domain D then its real and imaginary parts are harmonic in D.

Q.2. (a) Show that a mapping $w = f(z) = \cos(z)$ is conformal at points $z_1 = i$, $z_2 = 1$, $z_3 = \pi + i$ and determine the angle of rotation by $\alpha = Arg(f'(z_0))$ at all points.

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0, \\ 0, & z = 0, \end{cases}$$

Cauchy Riemann equations are satisfied at origin but $f'(0)$ fails to exist.

Q.3. (a)Does Converse of Caushy-Gorsat theorem holds in general. State and prove Morera's theorem.
(b) Show that $f(z) = \cosh(z + \frac{1}{z})$ can be expanded as a Laurent's series $\sum\limits_{n=-\infty}^{\infty} a_n z^n$, where

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta \, d\theta.$$

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UNIVERSITY OF THE PUNJAB

Part-I **A/2016**
Examination:- M.A./M.Sc.

Roll No. ....

Subject: Mathematics (Old & New Course) **TIME ALLOWED: 3 hrs.**
PAPER: V (Topology and Functional Analysis) **MAX. MARKS: 100**

NOTE: Attempt FIVE questions in all, selecting at least TWO from each section.

Section I

Q1.) *a)*

(1) Let A be a subset of $(\mathbb{R}, \tau)$, where τ is the usual topology. Find the interior and closure of $A = (0,1] \cup \{2,3\}$. (5)

(2) Consider $X = \{x, y, z\}$ and $Y = \{1,2,3\}$ with topologies $\tau_X = \{\varphi, X, \{x\}, \{x,y\}, \{x,z\}\}$ and $\tau_Y = \{\varphi, Y, \{1,2\}\}$, respectively. Check whether the function $f : X \to Y$ defined by $f(x) = 2, f(y) = 1$, and $f(z) = 2$ is continuous or not. (5)

b) Let A be a subset of topological space X. Prove that a function $f : X \to Y$ is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$. (10)

Q2). *a)* Prove that the collection $\mathcal{B}$ of open disks is a basis for the usual topology on $\mathbb{R}^2$. (10)

b) Prove that the property of being a completely regular space is heriditary. (10)

Q3). *a)* Prove that every closed subspace of T_4-space is a T_4-space. (10)

b) Prove that a compact subset of $\mathbb{R}^n$ is closed and bounded. (10)

Q4). *a)* Prove that a topological space X is connected if and only if there does not exist a surjective continuous function from X to a two-point discrete space. (10)

b) Prove that a continuous image of a compact space is compact. (10)

Section II

Q5.) *a)* Prove that a subspace Y of a complete metric space is complete if and only if Y is closed. Also deduce that $[0,1]$ is a complete subspace of $\mathbb{R}$. (10)

b) Prove that the unitary space $\mathbb{C}^n$ is complete. (10)

Q6). *a)* Let A be a subset of a metric space (X,d). Then prove that $|d(x,A) - d(x,B)| \leq d(x,y)$. (10)

b) Define convex set. Give an example of convex set and prove that $B(x_0; r) = \{x \in N \mid \|x - x_0\| < r\}$ is convex in a normed space N. (10)

Q7). *a)* Let S be a closed subspace of a Banach space N. Then Show that N/S is Banach space with the norm defined by $\|x + S\|_1 = \inf_{x \in S} \|x + s\|$. (10)

b) Prove that every finite dimensional subspace M of a normed space N is complete. In particular, every finite dimensional normed space is complete. (10)

Q8). *a)* Let N, M be a Normed spaces. Prove that if M is Banach space then space $B(N,M)$ of all bounded linear operators is a Banach space under the norm $\|T\| = \sup_{\|x\|=1} \|Tx\|, x \in N$. (10)

b) Prove that any two norms defined on a finite dimensional normed space are equivalent. (10)

Q9). *a)* Let $\{x_n\}$ and $\{y_n\}$ be any two sequences in an inner product space V. Prove that if $x_n \to x$ and $y_n \to y$, then $\langle x_n, y_n \rangle \to \langle x, y \rangle$. (10)

b) Let H be a Hilbert space and $y \in H$ then the function $f_y : H \to F$ given by $f_y(x) = \langle x, y \rangle$ defines a linear functional on H. Moreover, $\|f_y\| = \|y\|$, $y \in H$. (10)

UNIVERSITY OF THE PUNJAB

Part-I A/2016
Examination:- M.A./M.Sc.

Roll No. ....

Subject: Mathematics (Old & New Course)
PAPER: IV (Mechanics)

TIME ALLOWED: 3 hrs.
MAX. MARKS: 100

NOTE: Attempt any FIVE questions selecting at least TWO questions from each section. Each question carries equal marks.

SECTION I

1. (a) Evaluate $\int_1^2 \mathbf{A}(t)\cdot\mathbf{B}(t)\times\mathbf{C}(t)\,dt$ for $\mathbf{A}(t) = t\mathbf{i} - 3\mathbf{j} + 2t\mathbf{k}$, $\mathbf{B}(t) = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{C}(t) = 3\mathbf{i} + t\mathbf{j} - \mathbf{k}$.

   (b) Give a definition of $\iint_S \mathbf{F}.\mathbf{n}ds$ over a surface S in terms of limit of a sum, where F is a vector function. Suppose that the surface S has projection R on the xy-plane. Show that $\iint_S \mathbf{F}.\mathbf{n}ds = \iint_R \mathbf{F}.\mathbf{n}\frac{dxdy}{|\mathbf{n}.\mathbf{k}|}$.

   [10+10=20]

2. (a) Show that a necessary and sufficient condition that $F_1dx + F_2dy + F_3dz$ be an exact differential is that $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is irrotational. Show that $(y^2z^3\cos x - 4x^3z)\,dx + (2z^3y\sin x)\,dy + (3y^2z^2\sin x - x^4)\,dz$ is an exact differential of a function ϕ or not, if so then find ϕ.

   (b) What are Green's identities? Prove that $\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)dV = \iint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot d\mathbf{S}$.

   [10+10=20]

3. (a) Find the expression for Curl A and $\nabla^2\psi$ in orthogonal curvilinear coordinates and apply this result to find these quantities in spherical coordinates.

   (b) Let (u_1, u_2, u_3) and $(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ be the two general curvilinear coordinate systems. Find the relation between the contravariant components of a vector field A in the two coordinate systems.

   [10+10=20]

4. (a) Define conjugate metric tensor g^{ij} of a metric tensor g_{ij}. Show that for an orthogonal coordinate system $g^{11} = 1/g_{11}$, $g^{22} = 1/g_{22}$, $g^{33} = 1/g_{33}$. Find g and g^{ij} for the metric given by $ds^2 = 5(du^1)^2 + 3(du^2)^2 + 4(du^3)^2 - 6du^1du^2 + 4du^2du^3$.

   (b) Determine the Christofell symbols of first and second kind in (i) cylindrical and (ii) spherical coordinates.

   [10+10=20]

5. (a) Covariant components of a tensor in rectangular coordinates are $2x - z$, x^2y, yz. Find its contravariant components in spherical coordinates r, θ, φ.

   (b) Differentiate between symmetric and skew symmetric tensors. What is the largest number of different components which a symmetric contravariant tensor of rank two can have if (a) $N = 4$, (b) $N = 6$? What is the number for any value of N?

   [10+10=20]

SECTION II

6. (a) Find the expression for the third derivative of the position vector $\mathbf{r}(t)$ of a particle moving in a rotating coordinate system $Oxyz$ with respect to a fixed coordinate system $OXYZ$.

   (b) An xyz−coordinate system rotates about the z−axis with angular velocity $\omega = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ relative to a fixed coordinate system where t is the time. The origin of the xyz−system has position vector $\mathbf{R} = t\,\mathbf{i} - \mathbf{j} + t^2\,\mathbf{k}$ with respect to XYZ−system. Find the apparent and true acceleration of the particle at any time t whose position vector is given by $\mathbf{r} = (3t + 1)\,\mathbf{i} - 2t\,\mathbf{j} + 5\,\mathbf{k}$.

   [10+10=20]

7. (a) What is moment of inertia? What does it represent? Find the moment of inertia of a solid sphere about its diameter.

   (b) State and prove perpendicular axis theorem for a continuous distribution of mass in a plane. Apply this result to find the moment of inertia of an elliptic disc of semi-major and semi-minor axes of lengths a and b, about an axis perpendicular to the plane of elliptic disc passing through its center.

   [10+10=20]

8. (a) Find the expression for moment of inertia of a rigid body about an arbitrary line making angles α, β, γ with the x, y and z coordinate axes.

   (b) What are principal moments of inertia for a rigid body and the directions of the principal axes? Find the principal moments of inertia and directions of the principal axes of a uniform solid cylinder of radius a.

   [10+10=20]

9. (a) Derive Euler's equations of motion and establish the constancy of kinetic energy and angular momentum. Write Euler's equations of motion in case the axes are not principal axes.

   (b) Set up equations for the motion of a spinning top having one point fixed on its axis, express these equations in Euler angles and find the condition for steady precession of a top.

   [10+10=20]