UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Supply 2020 & Annual – 2021

Roll No. ....

Subject: Mathematics — Paper: I (Advanced Analysis) — Time: 3 Hrs. — Marks: 100

NOTE: Attempt FIVE questions in all selecting at least TWO questions from each Section.

SECTION – I

Q1.	a) Show that cartesian product $P \times P$ is denumerable, where $P = \{1,2,3, \dots \dots \dots\}$. b) State and prove Bernstein Theorem.	10 10
Q2	a) For cardinal numbers α, β, γ if $\alpha \leq \beta$, then show that $\alpha\gamma \leq \beta\gamma$ b) Suppose $f: S \to T$ is an ordered isomorphism between two ordered sets S and T. Then show that $a \in S$ is a first, last, minimal or maximal elements of S if and only if $f(a)$ is a first, last, minimal or maximal elements of T.	10 10
Q3	a) Let A be a well-ordered set and $S(A)$ be the collection of all initial segments of elements in A, then Show that A is ordered isomorphic to $S(A)$. b) Let A be a well-ordered set and S be a subset of A with the following property; If $a \preccurlyeq b$ and $b \in S$, then $a \in S$. Then show that $A = S$ or S is an initial segment of A.	10 10
Q4	a) Show that for ordinal numbers λ, μ and η, $(\lambda\mu)\eta = \lambda(\mu\eta)$ and $1.\lambda = \lambda$ b) Show that the set of natural number equipped with divisibility is partial ordered set.	10 10

SECTION – II

Q5	a) Show that Lebesgue measure of countable set is zero. b) For any set A and $\epsilon > 0$, show that there is an open set O such that $A \subseteq O$ and $m^*(O) < m^*(A) + \epsilon$.	10 10
Q6	a) Let $\{E_n\}$ be a decreasing sequence of measurable sets and $m(E_1) < \infty$, then show that $m(\bigcap_{i=1}^{\infty} E_i) = m(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n)$. b) For a subset $E \subseteq \mathbb{R}$ and $\epsilon > 0$ there is a closed set $F \subseteq E$ such that $m^*(E \backslash F) < \epsilon$ if and only if E is measurable.	10 10
Q7	a) Show that the interval [0, 1) contains a non-measurable set. b) Show that every Borel set is measurable. Does converse hold? Justify your answer. c) Show that the characteristic function χ_A defined on measurable set is measurable if and only if A is measurable subset of D.	8 6 6
Q8	a) Let f and g be extended real- valued measurable function which are finite almost everywhere then show that $f+g$ is measurable. b) Show that an extended function f is Borel measurable if and only if for ant open set V in $\bar{R}, f^{-1}(V)$ is a Borel set. c) Let f be a non-negative measurable function on E. Then show that $\int f = 0$ if and only if f=0 a.e on E.	8 6 6
Q9	a) State and prove Fatou's Lemma. b) Let E be measurable se, $1 \leq p \leq \infty$ and q be the conjugate of p. For $f \in L^p(E)$ and $g \in L^q(E)$, show that $f.g$ is integrable and $\int_E \lvert f.g \rvert \leq \lVert f \rVert_p . \lVert g \rVert_q$. Also show that if $f \neq 0$, the function $f^* = \lVert f \rVert_p^{1-p} sgn(f) \lvert f \rvert^{p-1} \in L^q(X,\mu)$ such that $\int_E f.f^* = \lVert f \rVert_p$ and $\lVert f^* \rVert_q = 1$ c) Define truncation function and calculate Lebesgue integral of $h(x) = \frac{1}{\sqrt{x}}$ on [0,1]	6 8 6

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M.A./M.Sc. Part – II Supply 2020 & Annual – 2021

Subject: Mathematics | Paper: II (Methods of Mathematical Physics) | Time: 3 Hrs. | Marks: 100

NOTE: Attempt FIVE questions in all selecting at least TWO questions from each Section.

SECTION – I

1(a)	Show that the eigen functions corresponding to periodic SL system are orthogonal.	10
1(b)	Suppose that $u(x)$ and $v(x)$ are two solutions of the SL systems then prove that the Lagrange's identity must hold $uL(v)-vL(u)=\frac{d}{dx}\{p(x)[u(x)v'(x)-u'(x)v(x)]\}$	10
2(a)	Use the method of Frobenius to find a series solution in x of the DE $2xy''+y'-y=0$	10
2(b)	Find linearly independent solutions of the DE $y''-2xy'+y=0$ by the method of power series expansions.	10
3(a)	Solve the boundary value problem $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, 0\le x\le a, 0\le y\le b, u_x(0,y)=0, u_x(a,y)=0, u_y(x,0)=0, u(x,b)=f(x)$	10
3(b)	Find the complete integral surface of the PDE $x(y^2+z)p-y(x^2+z)q=(x^2-y^2)z$ containing the straight line $x+y=0, z=1$	10
4(a)	Find the integral surface of the PDE $(y^2-z^2)z_x-xyz_y=xz$ containing the curve $x=y=z, x>0$	10
4(b)	Prove the integral representation $F_{21}(a,b,c;x)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-a-1}(1-xt)^{-a}\,dt$	10
5(a)	Discuss the orthogonality of Bessel functions and show that $\int_a^b xJ_\mu(\alpha x)J_\mu(\beta x)dx=0$	10
5(b)	Prove the Rodrigues formula $P_n(x)=\frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n$	10

SECTION – II

6(a)	(i) Evaluate the inverse Laplace transform of $L^{-1}\left(\frac{e^{-3s}}{(s-2)^3}\right)$. (ii) Use convolution theorem to calculate the Laplace transform of $f(t)=\int_0^t (t-\beta)^3 e^\beta \sin\beta\, d\beta$	10
6(b)	Solve the DE by Laplace transform method $y''(t)-2ky'(t)+k^2y(t)=f(t)$	10
7(a)	Calculate Fourier sine transform of the function $f(x)=e^{-x}\cos x$	10
7(b)	Use Fourier transform to solve the potential equation $u_{xx}+u_{yy}=0$ for the potential function $u(x,y)$ in the semi-infinite strip $0<x<c, y>0$ that satisfies the conditions $u(0,y)=0, u_y(x,0)=0, u_x(c,y)=f(y)$	10
8(a)	Construct the Green's function associated with the boundary value problem $x\frac{d^2u}{dx^2}+\frac{du}{dx}+\lambda r(x)u=0$ with $u(1)=0$ and $u(0)$ finite.	10
8(b)	Construct the Green's function associated with the boundary value problem $u''-u+\lambda u=0$ with $u(0)=0$ and $u(1)=0$.	10
9(a)	Find the extremal of the problem $I[y]=\int_0^1 (x+y'^2)dx, y(0)=1, y(1)=2$	10
9(b)	Discuss geodesic problem. A uniform cable is fixed at its ends at the same level in space and is allowed to hang under gravity. Find the final shape of the cable.	10

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M.A./M.Sc. Part – II Supply 2020 & Annual – 2021

Subject: Mathematics   Paper: III (Numerical Analysis)   Time: 3 Hrs.   Marks: 100

NOTE: Attempt FIVE questions in all selecting at least TWO questions from each Section.

SECTION – I

Q1. (8+12)

(a) Write an algorithm to find an approximate root of the non-linear equation $f(x)=0$ using secant method.

(b) Find a real root of the equation $x^3 + x^2 - 100 = 0$ by fixed point iterative method.

Q2. (10+10)

(a) Solve the system of equations by triangularisation method

$$10x_1 + x_2 + x_3 = 12$$
$$2x_1 + 10x_2 + x_3 = 13$$
$$x_1 + x_2 + 5x_3 = 7$$

(b) Find the Dominant Eigen value and corresponding Eigen vector of the following matrix by power method

$$\begin{bmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{bmatrix}$$

Q3. (10+10)

(a) Solve $\frac{dy}{dt} = 1 - 2ty; y(0) = 0, h = 0.2 \; for \; t = 0.2, 0.4, 0.6$ using Modified Euler's method.

(b) Use Runge-Kutta method of order four to solve the differential equation $\frac{dy}{dt} = \exp(-2t) - 2y$ over $[0, 0.2]$ with $y(0) = \frac{1}{10}$ by taking $h = 0.1$

Q4. (10+10)

(a) Solve the following system by Crout's method

$$10x_1 - 7x_2 + 3x_3 + 5x_4 = 6$$
$$-6x_1 - 8x_2 + x_3 + 4x_4 = 5$$
$$3x_1 - x_2 + 4x_3 + 11x_4 = 2$$
$$5x_1 - 9x_2 + 2x_3 + 4x_4 = 7$$

(b) Prove that

(i) n-th difference of n-th degree polynomial is constant and (n+1)-th difference is zero.

(ii) $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

Q5. (10+10)

(a) Derive the Adams-Bashforth predictor and corrector formulae to solve the first order initial value problems.

(b) Solve the following system of differential equations using Taylor's series method of order 3 for x=0.3, 0.6

$$\frac{dy}{dt} = 1 + tz, \frac{dz}{dt} = -ty, y(0) = 0, z(0) = 1.$$

P.T.O

SECTION – II

Q6. (10+10)

(a) Using Newton's divided difference interpolation formula find $f(6)$ and $f(12)$

x	4	5	8	11	13	14
y	46	88	175	294	343	451

(b) Interpolate by means of Gauss's backward interpolation formula, find the sales of a person for the year 1968 from the following data

year	1940	1950	1960	1970	1980	1990
y	3	11	31	69	131	223

Q7. (10+10)

(a) Find the global error of composite Simpson's 3/8 rule.

(b) A solid of revolution is formed by taking about the x-axis, the area between x-axis and line x=0 and x=1 and a curve through the points with the following coordinates

x	0	0.25	0.5	0.75	1
y	1	0.98567	0.95893	0.91436	0.85659

Q8. (10+10)

(a) Find the first derivative of Y at X=35 using Stirling's formula from the following data

X	10	20	30	40	50
Y	0.1023	0.1047	0.1971	0.1096	0.1122

(b) Find the first and second derivatives of $f(x)$ at $x = 1.9$ from the following data

x	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$	0	3.1234	4.1971	6.2396	7.4322	8.1148

Q9. (10+10)

(a) Solve the following difference equation:

$$y_{n+2} - 3y_{n+1} + 4y_n = \sin 6n + \cos 6n + 7.$$

(b) Solve the following difference equation

$$y_{n+2} - 9y_{n+1} - 52y_n = 13^n(-3n^2 + 1).$$

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M.A./M.Sc. Part – II Supply 2020 & Annual – 2021

Subject: Mathematics — Paper: IV-VI (Opt. i) (Mathematical Statistics) — Time: 3 Hrs. Marks: 100

NOTE: Attempt FIVE questions in all selecting at least TWO questions from each Section.

SECTION – I

Q.1 **(a)** If A, B and C are pairwise independent events in a sample space S and A is independent of $B \cup C$. Then prove that A, B and C are mutually independent. **(10)**

(b) Ten vegetables cans, all the same size, have lost their labels. It is known that 5 contains tomatoes and 5 contains corn. If 5 are selected at random, what is the probability that all contain tomatoes? What is the probability that 3 or more contain tomatoes? **(10)**

Q.2 **(a)** State and prove reproductive property of Poisson distribution. **(10)**

(b) Write down chief characteristics of hyper-geometric, negative binomial, uniform and gamma distribution. **(10)**

Q.3 **(a)** State and prove Chebyshev's inequality. **(10)**

(b) A coin is tossed 200 times. Find the probability of getting **(10)**

i) between 80 and 120 heads inclusive
ii) less than 90 heads.
iii) Exactly 100 heads.

Q.4 **(a)** If $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution with mean μ and standard deviation σ, then obtain the sampling distribution of **(10)**

$\bar{X} = \frac{\sum x}{n}$ and $s^2 = \frac{\sum(x-\bar{x})^2}{n-1}$.

(b) In a normal distribution with $\mu = 47.6$ and $\sigma = 16.2$, find **(10)**

i) the probability that a single observation will be larger than 50,
ii) two points such that a single observation has a 97% probability of falling between them,
iii) P_{10}, P_{30} *and* P_{99}

SECTION – II

Q.5 **(a)** If X_r and X_s are the r^{th}' and s^{th}' random variables of random sample of size n drawn from the finite population $\{C_1, C_2, \ldots, C_N\}$. Then **(10)**

$$Cov(X_r, X_s) = \frac{\sigma^2}{N-1}.$$

(b) If X has the standard normal distribution, find the probability density of $Z = X^2$. **(10)**

Q.6 **(a)** Determine multiple regression equation in terms of linear correlation coefficients. **(10)**

(b) Given the joint density **(10)**

$$f(x,y) = \begin{cases} 2 & \text{for } 0 < y < x < 1 \\ 0 & \text{elsewhere} \end{cases}.$$

Show that $\mu_{Y/x} = \frac{x}{2}$ and $\mu_{X/y} = \frac{1+y}{2}$.

Q.7 **(a)** Let the random variable X have marginal density $f_1(x) = 1,\quad \frac{-1}{2} < x < \frac{1}{2}$ and let the conditional density of Y be **(10)**

$$f(y|x) = 1, x < y < x + 1 \qquad \frac{-1}{2} < x < 0$$

$$= 1, -x < y < 1 - x \qquad 0 < x < \frac{1}{2}$$

Show that random variables X and Y are uncorrelated.

(b) Compute the coefficient of determination of the following data **(10)**

X	5	11	4	5	3	2
Y	31	40	30	34	25	20

Q.8 **(a)** Let $X_1, X_2, \ldots, X_n$ be a random sample of size n taken from a normal population with mean μ and varianceσ^2. If $\bar{x}$ and S^2 represent the mean and biased variance of the sample chosen above. Then prove that $\frac{nS^2}{\sigma^2}$ is a Chi-square variate with $n-1$ degree of freedom. **(10)**

(b) If $F \sim F(v_1, v_2)$ then $Y = (1 + \frac{v_1}{v_2}F)^{-1} \sim \beta(\frac{v_1}{2}, \frac{v_2}{2})$. **(10)**

Q.9 **(a)** If n denotes the degrees of freedom of a t-distribution, then show that **(10)**

$$(n - 2r)\,\mu'_{2r} = n(2r - 1)\,\mu'_{2r-2}$$

Where μ' represents moments about origin.

(b) Find moment generating function for χ^2-distribution. Use it to evaluate mean and variance of the distribution. **(10)**

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Subject: Mathematics | Paper: IV-VI (Opt. ii) [Computer Applications] | Time: 3 Hrs. | Marks: 50

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

SECTION – I

Q.1 (10)

Write a program to find the inverse of a square matrix using Function Subprogram.

Q2. (10)

Write a program to find the roots of a quadratic equation using arithmetic IF-STAMENT.

Q3. (10)

write a program to find the complex conjugate of a complex ~~number.~~

Q4. (10)

Write a program to find the largest and smallest numbers and their locations in a list of 100 numbers.

SECTION – II

Q5. (10)

Write a program, to print the values of $y(0.1), y(0.2), z(0.1)$ and $z(0.2)$ from

$$\frac{dy}{dx} = x + z,\ \frac{dz}{dx} = x - y^2 \text{ with } y(0) = 2,\ z(0) = 1$$

by improved Euler's method.

Q6. (10)

Write a program to find a positive root of $e^{-3x} - \cos(\frac{\pi x}{4}) = 0$, correct to four decimal places using the Bisection method.

Q7. (10)

Write a program to solve the following system of equations using Gauss Seidel iterative method:

$$11x - 303y + 200z = 205,\ 4x + 11y - z = 33,\ 6x + 3y + 12z = 35$$

Q8. (10)

From the following table write a program to find $f(0.05)$ using Newton's Forward difference formula.

Table

x	-1	0	1	2	3
$f(x)$	2	1	2	9	28

Q9. (10)

Write the Mathematica statements for the following:

1. Plot the following functions, $Sin3x + 0.5$, $Cos2x + 0.2$ and $Secx$ in the same window over the interval, $-\frac{\pi}{2} < x < \frac{+\pi}{2}$
2. Find all the solutions of $x^4 + x^3 - 8x^2 - 5x + 15 = 0$ which are greater than 2
3. Solve for $x : e^{2x} + e^x = 3$
4. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 7 & 2 \\ 1 & 3 & 2 \end{bmatrix}$ find $A+B,\ A-B,\ AB,\ A^{-1}, B^{-1}$
5. Compute the first five derivatives of $f(x) = e^{x^2}$ at $x = 0$

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Subject: Mathematics Paper: IV-VI (opt.xi) [Theory of Approximation & Splines] Time: 3 Hrs. Marks: 100

NOTE: Attempt any FIVE questions selecting at least TWO questions from each section.

	SECTION-I	
	QUESTION No. 1	**Marks**
(a)	Derive the relation for the reflection of point $P(x,y)$ about the line $y = x$ and $y = -x$.	(10)
(b)	Discuss the scaling transformation in 2D.	(10)
	QUESTION No. 2	
(a)	Derive the relation for the rotation of the point $P(x,y)$ about the point $Q(x_p, y_p)$.	(10)
(b)	Find the image of the circle $x^2 + y^2 = 4$, under the transformation of shear parallel to x-axis by the factor 2.	(10)
	QUESTION No. 3	
(a)	Show that the trigonometric representation of Chebyshev polynomial on $[-1,1]$ is $T_N(x) = cos(Narccos(x))$.	(10)
(b)	Find the least-squares parabola for the four points $(-3,3)$, $(0,1)$, $(2,1)$ and $(4,3)$.	(10)
	QUESTION No. 4	
(a)	Establish the following Padé Approximation: $$e^x \approx R_{2,2}(x) = \frac{12 + 6x + x^2}{12 + 6x + x^2}$$	(10)
(b)	Find the power fits $y = \frac{A}{x}$ and $y = \frac{B}{x^2}$ for the following data and use $E_2(f)$ to determine which curve fits best. x_k: 0.5, 0.8, 1.1, 1.8 y_k: 7.1, 4.4, 3.2, 1.9	(10)
	SECTION-II	
	QUESTION No. 5	
(a)	State and prove de Casteljau algorithm for Bernstein-Bézier curve of degree n.	(10)
(b)	Derive the formula for barycentric coordinate with respect to a triangle $\Delta V_1V_2V_3$, $V_i = (x_i, y_i), i = 1,2,3.$	(10)
	QUESTION No. 6	
(a)	Prove that rational quadratic Bernstein-Bézier curve represents conic section.	(10)
(b)	Prove that the image of Bernstein-Bézier curve $P(\theta)$ under any affine transformation $\emptyset$ is $\emptyset(P(\theta)) = \sum_{i=0}^{n} B_i^n(\theta)\,\emptyset(b_i)$.	(10)
	QUESTION No. 7	
(a)	Define tensor product surface. Construct the Bernstein Bezier cubic patch.	(10)
(b)	Derive the recursive relation for Bernstein polynomials.	(10)
	QUESTION No. 8	
(a)	Derive the relation for the error bound of cubic Hermite interpolation.	(10)
(b)	Prove that a natural spline of degree $2n - 1$ with knots at the points $a = x_0 < x_1 < \cdots < x_k = b$ has the following representation in terms of truncated power function over the interval $[a, b]$ $$S(x) = \sum_{i=0}^{n-1} a_i x_i + \sum_{i=0}^{k} c_i\,(x - x_i)_+^{2n-1}$$	(10)
	QUESTION No. 9	
(a)	Define integral B-spline, uniform-spline, periodic B-spline, closed periodic B-spline.	(10)
(b)	If $N_0^{(2)}(t) = \begin{cases} t & 0 \le t \le 1 \\ 2 - t & 1 \le t \le 2 \\ 0 & otherwise \end{cases}$ Determine $N_{-5}^{(2)}(t)$ using direct method.	

The data table of Question 4(b):

x_k	0.5	0.8	1.1	1.8
y_k	7.1	4.4	3.2	1.9

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M.A./M.Sc. Part – II Supply 2020 & Annual – 2021

Subject: Mathematics Paper: IV-VI (Opt. iii) (Group Theory) Time: 3 Hrs. Marks: 100

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

SECTION-I

Q 1. (a) Define a Sylow p-subgroup of a group G. Prove that evry group of finite order n contains a Sylow p-subgroup where p is prime and divides n. Deduce that in a group of order 80, the Sylow 2-subgroup (if exists) must be of order 16.

(b) Let G be a finite group. Prove that if every maximal subgroup of G is normal in G then every Sylow- p subgroup of G is normal in G.

Q 2. (a) Classify all groups of order at most 8.

(b) State and prove Orbit-Stablizer Theorem.

Q 3. (a) Let $G = A \times B$ be the direct product of A, and B. Prove that $Z(G) = Z(A) \times Z(B)$ where $Z(G)$, $Z(A)$, $Z(B)$ are centers of G, A and B respectively Deduce that a group of order 121×169 is always abelian.

(b) Define Holomorph of a group. Find all normal (semi direct) products of C_3 by C_2.

Q 4. (a) The group $S_3 \times Z_2$ is isomorphic to one of the following groups:
$Z_{12,}$ $Z_6 \times Z_2$, A_4 or D_6. Determine the group by elimination.

(b) Define Characteristic and Fully Invariant subgroups. Let C be a characteristic suhgroups of a normal subgroup N of a group G. Prove that C is normal in G.

SECTION-II

Q 5. (a) Define a Normal Series in a group G. Prove that any two normal series of a group G have isomorphic refinements.

(b) Define a composition series in a group G. Prove that a group G has a composition series if and only if its all ascending and descending normal chains break off.

Q 6. (a) Prove that the direct product of solvable groups is solvable.

(b) Prove that every finite group-p group is solvable.

Q 7. (a) Define a nilpotent group. Prove that every subgroup and factor group of a nilpotent group is nilpotent. Give a counter example to show that the converse is not true in general.

(b) Prove that every finite group-p group is nilpotent.

Q 8. (a) Show that a normal subgroup H of G is contained in the Frattini subgroup of G if and only if H has no partial complement in G.

(b) Prove that $GL(n,q)/SL(n,q)$ is isomorphic to the group $GF(q)^{*} = F_q^{*}$

Q 9. (a) Let G be an extension of N by H and $\{s(h)\}$ be a section of G through H with sectional factor set $f : H \times H \to N$ prove that,

(i) $f(1,h) = 1 = f(h,1), h \in H$

(ii) $f(h_1,h_2)f(h_1h_2,h_3) = f(h_2,h_3)^{s(h_1)} f(h_1,h_2h_3), h_i \in H, i = 1,2,3$

(b) For an odd prime p, write the complete list of groups of order $p, 2p, p^2$ and $p^3, p < 31$.

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Subject: Mathematics | Paper: (IV-VI) (Opt. iv) [Rings & Modules] | Time: 3 Hrs. | Marks: 100

NOTE: Attempt any FIVE questions selecting at least TWO questions from each section.

Section I

Q 1. a) Let R be an integral domain such that $R[x]$ is a principal ideal domain, then show that R is a field.

b) Define Euclidean Domain. Show that the polynomial ring $F[x]$ is Euclidian domain, where F is a field. 10+10

Q 2. a) Calculate HCF for the polynomials $x^3 + 2x^2 + 4x - 7$ and $x^2 + x - 2$ in $\mathbb{Q}[x]$.

b) Prove that $R = \{a + b\sqrt{-5} : a, b \in Z\}$ is not a unique Factorization Domain. 10+10

Q 3. a) If $p, q \in R[x]$, then

i. $\deg(p + q) \leq \max\{\deg(p), \deg(q)\}$

ii. $\deg(pq) \leq \deg(p) + \deg(q)$

iii. If R is an integral domain, then $\deg(pq) = \deg(p) + \deg(q)$

b) Let $\mathbb{C}$ be a field of complex numbers. Then every polynomial $p(x) \in \mathbb{C}[x]$ of degree $n \geq 1$ has n roots over $\mathbb{C}$ 10+10

Q 4. a) Find the smallest extension of $\mathbb{Q}$ having a root of $x^4 - x^2 + 2$ in $\mathbb{Q}[x]$.

b) Polynomial $x^2 + 1$ over $\mathbb{R}$ and $\mathbb{C}$ is not splitting field for $x^2 + 1$ over $\mathbb{Q}$. 10+10

Q 5. a) Define extension of a field. Find the smallest extension of $\mathbb{Q}$ having a root of $x^3 - 2 \in \mathbb{Q}[x]$.

b) If L is finite extension of K and $K$is a finite extension of F, then prove that L is finite extension of F and $[L:F] = [L:K][K:F]$. 10+10

Section II

Q 6. a) Let $\mathbb{Q}$ be the set of rational numbers. Then show that $\mathbb{Q}$ is not $FG - \mathbb{Z}$- module.

b) Show that any two cyclic R- modules are isomorphic if and only if they have same order ideal. 10+10

Q 7. a) Let M be an R- module, where R is a commutative ring with identity. Show that M is simple if and only if $M \cong R/_I$, I is a maximal ideal of R.

b) Let R be a ring. Show that $R[x]$ is a $FG - R$-module if and only if $R = \{0\}$. 10+10

Q 8. a) Let M and Nbe two R-modules, $f: M \to N$ and $g: N \to M$ be two module homomorphisms such that $gof = I_M$ (identity map on M). Show that

$$N = Kerg \oplus Imf$$

b) Let A and C be submodules of an R- module M such that $A \subseteq C$. Then prove that $\frac{M/A}{C/A} \cong \frac{M}{C}$. 10+10

Q 9. Let R be a principal ideal domain and F be a free R-module of finite ranks. Prove that every submodule of F is free of rank $\leq$ s. 20

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Subject: Mathematics　　Paper: IV-VI (Opt. v) (Number Theory)　　Time: 3 Hrs.　Marks: 100

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

1 (a) State and Prove "Euclidean's Theorem". (10)

(b) Show that $(a-b)/(a^n - b^n)$ for all positive integers n (10)

2 (a) Solve the linear congruence $17x \equiv 11(mod 43)$ (10)

(b) Solve the system of linear congruences by Chinese Remainder Theorem. (10)

$$x \equiv 2(mod\,5)$$
$$x \equiv 3(mod\,7)$$
$$x \equiv 5(mod\,11)$$

3 (a) State and Prove "Fermat's Little Theorem". (10)

(b) Show that $5^{38} \equiv 4(mod\,11)$ (10)

4 (a) Prove that (I) if $\phi(n-1) = n-1 \Leftrightarrow n$ is prime. (10)

(ii) if $n = p^\alpha$, p is prime, then $\phi(n) = p^\alpha - p^{\alpha-1} = p^\alpha\left(1-\frac{1}{p}\right)$

(b) Prove that for each integer $n \geq 1$ then (10)

$$\sum_{d/n} \mu(d) = \begin{cases} 1 & \text{if} \quad n=1 \\ 0 & \text{if} \quad n>1 \end{cases}$$

5 (a) Prove that if a is a primitive root mod m then $1, a, a^2, ----, a^{\phi(m)-1}$ is a reduced residue system modulo m (10)

(b) Find all primitive roots of 7^2 (10)

SECTION-B

6 (a) Prove that an integer a is a quadratic residue $(mod\,p)$ if and only if $a^{\frac{p-1}{2}} \equiv 1(mod\,p)$ (10)

(b) Evaluate $\left(-\frac{168}{11}\right)$ (10)

7 (a) State and prove "Fermat's Last Theorem". (10)

(b) Prove that $g(x)$ and $f(x)$ are two non-zero polynomials over F are relativity prime over F, then there exist polynomials $s_0(x)$, $t_0(x)$ over F such that

$$1 = s_0(x)f(x), + t_0(x)g(x)$$ (10)

8 (a) Prove that if θ is algebraic over F. It has a unique minimal polynomial. (10)

(b) If p is prime number, then show that $\frac{x^p-1}{x-1}$ is an irreducible over R. (10)

9 (a) Prove that every α of $F[\theta]$ can be uniquely written in the form

$$\alpha = a_0 + a_1\theta + a_2\theta^2 + --- + a_{n-1}\theta^{n-1} = r(\theta)$$

where a_i are in F and n is degree of θ over F (10)

(b) For $\alpha, \beta \in R[\theta]$ then if $N_\alpha = 1$ if and only if α is unit. (10)

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M.A./M.Sc. Part – II Supply 2020 & Annual – 2021

Roll No. ....

Subject: Mathematics Paper: IV-VI (opt.vi) (Fluid Mechanics) Time: 3 Hrs. Marks: 100

NOTE: Attempt any FIVE questions by selecting atleast TWO questions from each section. All questions carry equal marks.

Section - I

Q.1(a)	A boy weighs *1000lbf* when exposed to a standard earth gravity $g=32.174ft/s^2$. (a) What will the weight of this body be in Newton if it is exposed to the moon's standard acceleration $g_{moon}= 1.62\ m/s^2$? (b) How fast will the body accelerate if a net force of 400lbf is applied to it on the moon or on the earth?	(10)
(b)	Derive the standard equation of motion for an ideal fluid.	(10)
Q.2(a)	Explain the difference between Eulerian and Lagrangian description in fluid mechanics. Also explain which one in better in fluid mechanics with solid reasons?	(10)
(b)	Given the steady two-dimensional velocity distribution $u=Kx$, $v=-Ky$, $w=0$, where K is a positive constant, compute and plot streamlines of the flow, including directions, and give some possible interpretations of the pattern.	(10)
Q.3(a)	A tank *20 ft* deep and *7 ft* wide is layered with *8 ft* of oil, *6 ft* of water, and *4 ft* of mercury. Compute the total hydrostatic force and the resultant center of pressure of the fluid on the right-hand side of the tank.	(10)
(b)	What is law of conservation of mass? Which equation is derived with this law? Derive the equation in cylindrical coordinates.	(10)
Q.4(a)	Find the velocity potential if exists for the velocity field $u= a(x^2-y^2)$, $v=-2axy$, $w=0$.	(10)
(b)	The circle $x^2+y^2-2ay=0$ is situated in a two-dimensional shear flow with $u=Cy$, $v=0$. Find the circulation in the circle by (a) Direct calculation (b) Using Stokes' theorem.	(10)
Q. 5	What is a two-dimensional vortex? Considering a free vortex placed at origin, find (a) Strength of the vortex (c) Stream function (b) Complex velocity potential (d) Singularity of the flow	(20)

Section - II

Q.6(a)	What is non-dimensionalization in fluid mechanics? Explain the dynamical similarity.	(10)
(b)	Derive the stress – strain relationship for a viscous Newtonian fluid.	(10)
Q.7(a)	What is Joukowski aerofoil? Explain it applications with two examples.	(10)
(b)	For an incompressible axisymmetric vortex flow with $\vec{V} = -Ar\hat{e_r} + \frac{B}{r}\hat{e_\theta} + 2az\hat{e_z}$, calculate all normal and shear stresses	(10)
Q.8	How equation of motion for a viscous Newtonian fluid is derived? Derive it in Cartesian and Cylindrical coordinate systems.	(20)
Q.9(a)	Define generalized Couette flow, Formulate it and calculate the velocity field, average velocity, Maximum velocity and shearing stress.	(10)
(b)	The velocity distribution for the steady, laminar and fully developed flow between two fixed parallel plates is given by $u = -\frac{1}{2\mu}\frac{dp}{dx}(hy - y^2)$, *where* $\frac{dp}{dx}$ is the pressure gradient, h is the gap width between the plates and y is the distance measured upward from the lower plate. Determine the volumetric flow rate, maximum velocity and the shearing stress at both the plates.	(10)

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M.A./M.Sc. Part – II Supply 2020 & Annual – 2021

Roll No. ....

Subject: Mathematics | **Paper: IV-VI (opt.vii) [Quantum Mechanics]** | **Time: 3 Hrs.** | **Marks: 100**

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

Section-I

1. (a) Explain photo-electric effect and discuss the inability of classical mechanics to explain photo-electric effect. (10)

   (b) The work function of zinc is 3.6 eV. What is the energy of the most energetic photo-electron emitted by ultraviolet light wavelength $\lambda = 2500$ Å ($\AA = 10^{-10}$m and $h = 4.1357 \times 10^{-15}$eV/sec). (10)

2. (a) Let a_n and ϕ_n denote the eigen values and corresponding eigen fucntions of a Hermitian operator $\hat{A}$. Find the eigenvalues and eigenfunction of $\sin(\hat{A})$ and $\cos(\hat{A})$. (10)

   (b) At time $t = 0$, wave function of a particle is $\psi(x,0) = \begin{cases} A\frac{x}{a}, & 0 \le x \le a; \\ A\frac{b-x}{x-a}, & a \le x \le b; \\ 0, & \text{else,} \end{cases}$ where A, a and b are constants.

   - Normalize the wave function ψ.
   - What is the probability of finding the particle to the left of a?
   - Find $\langle x \rangle$.

   (10)

3. (a) A particle in the infinite square well has the initial wave-function (10)

$$\psi(x,0) = \sqrt{\frac{30}{a^5}}x(x-a), \quad 0 \le x \le a.$$

   Find $\psi(x,t)$ using energy eigenstates of the said particle.

   (b) Let $\hat{A}$ be a Hermitian operator. (10)

   - Show that eigenvalues of $\hat{A}$ are real.
   - Show that eigenfunction corresponding to distinct eigenvalues of $\hat{A}$ are orthogonal.

4. (a) Define parity operator. Find its eigenvalues and eigenfunctions. (10)

   (b) If operators $\hat{A}$ and $\hat{B}$ are compatible then show that they share common eigenstates. Further discuss the effect of degenerate eigenvalues on the common eigenstates of these operators. (10)

P.T.O

Section-II

5. (a) Derive eigenvalues and eigen functions for $\hat{L}^2$ and $\hat{L}_z$ angular momentum operators. (10)

   (b) Suppose there is a delta function bump in the center of the infinite square well problem, given by

$$H' = \alpha\delta(x - \frac{a}{2}),$$

   where α is a constant. Find the first order correction to the energies and corresponding eigenstates. (10)

6. (a) Assume that a particle has an orbital angular momentum with z component $\hbar m$ and square magnitude $\hbar^2 l(l+1)$. Show that (10)

   - $\hat{L}_+|l,m\rangle = \sqrt{(l-m)(l+m+1)}\hbar|l,m+1\rangle$
   - $\langle L_x^2\rangle = \langle L_y^2\rangle = \frac{\hbar^2 l(l+1) - m^2\hbar^2}{2}$

   (b) Solve Schrodinger wave equation for Hydrogen atom and find the allowed energy values of the electron. (10)

7. (a) Find scattering amplitude for 3-D spherically symmetric potential using Born's approximation. (10)

   (b) Prove the following statements. (10)

   - $[\hat{L}_+, \hat{L}_-] = 2\hbar\hat{L}_z$, where $\hat{L}_\pm$ are ladder operators.
   - If ϕ_m is an eigenfunction of $\hat{L}_z$ for eigenvalue $\hbar m$, then show that $\hat{L}_-\phi_m$ is also an eigenfunction of $\hat{L}_z$ corresponding to eigenvalue $\hbar(m-1)$.

8. (a) Let $|n\rangle$ be the n^{th} eigenfunction (Dirac notation) of quantum mechanical harmonic oscillator. If $\hat{a}$ and $\hat{a}^\dagger$ denote annihilation and creation operators for quantum mechanical harmonic oscillator, respectively, then find the expectation values of the following operators: (10)

   - $(\hat{a}\hat{a}^\dagger)^2$
   - $\hat{a}^2(\hat{a}^\dagger)^2$
   - $(\hat{a}^\dagger)^2\hat{a}^2$

   (b) Consider two non-interacting particles, both of mass m, in the infinite square well($0 \leq x \leq a$). Write down the Hamiltonian for these two non-interacting particles. Also find the ground eigenstates and corresponding energies for identical bosons and identical fermions.

9. (a) Derive expressions for $\hat{L}_x$ and $\hat{L}_y$ in spherical polar co-ordinates. (10)

   (b) Find the spectrum of energy for Hamiltonian $\hat{H}$ of rigid rotator using orbital angular momentum operator $\hat{L}$. Is the spectrum continuous or discrete? Justify your answer. Also show that the frequencies of photons due to energy decays between successive level of a rotator with moment of inertia I are given by $\hbar\omega = \frac{\hbar^2}{I}(l+1)$ or $\hbar\omega = \frac{\hbar^2}{I}(l)$. (10)

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M.A./M.Sc. Part – II Supply 2020 & Annual – 2021

Roll No. ....

Subject: Mathematics

Paper: IV-VI (Opt.viii) (Special Theory of Relativity and Analytical Dynamics) **Time: 3 Hrs. Marks: 100**

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

SECTION I

1. (A) State the fundamental postulates of special relativity. Derive the Lorentz transformation for the observers in two frames of reference moving relative to each other with uniform velocity along one of the coordinate axes. Find the expression for length contraction, time dilation and velocity addition formula.

   (B) Events that are simultaneous in one reference frame are not simultaneous in another reference frame moving with respect to the first. Use Lorentz transformation to justify your answer. [10+10=20]

2. (A) A rocket leaves the earth at a speed of $0.6c$. A second rocket leaves the first at a speed of $0.9c$ with respect to the first. Calculate the speed of the second rocket with respect to earth if (a) it is fired in the same direction as the first one, (b) it is fired in a direction opposite to the first.

   (B) What do you understand about the laboratory frame and center of mass frame. How are they related to each other? Find the expressions for the energy and momentum components in laboratory frame for the two particles initially independent from each other coming closer together.

   [10+10=20]

3. (A) Explain the structure of a null-cone and the regions namely the timelike, the spacelike, lightlike, future directed and past-directed regions.

   (B) For the Minkowski metric $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$, identify the vectors as time like, space like or light like: (a) $T^\mu = (1,0,0,0)$, (b) $X^\mu = (0,1,0,0)$, (c) $Y^\mu = (0,0,1,0)$, (d) $Z^\mu = (1,0,1,0)$. [10+10=20]

4. (A) Let us assume that a particle (projectile particle of mass m_P) moving with certain velocity strikes another particle (target particle of mass m_T). The two particles move off. Assume that the collision of two particles generates a new particle. Find the expression for minimum kinetic energy (threshold energy T_0) required for production of this new particle (of rest mass m_N).

   (B) What is a four-vector potential? Express Maxwells field equations in the four-vector form. [10+10=20]

5. (A) Derive the expression for $F^{\mu\nu}$, the Maxwell field tensor and ${}^*F^{\mu\nu}$, the dual field tensor. Find the expression for ${}^*F_{\mu\nu}F^{\mu\nu}$ in terms of **E** and **B** for the Minkowski metric $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$.

   (B) Derive Maxwell's equations from Maxwell's Tensor $F^{\mu\nu}$. [10+10=20]

SECTION II

6. (A) Let $\mathbf{r}_\nu = \mathbf{r}_\nu(q_\alpha)$, $\alpha = 1, 2, \ldots n$ be the position vector of ν-*th* particle in terms of generalized coordinates. Prove that the kinetic energy satisfies the relations: (a) $T = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta$, where $a_{\alpha\beta}$ are the functions of generalized coordinates q_α and (b) $\dot{q}_1\frac{\partial T}{\partial \dot{q}_1} + \dot{q}_2\frac{\partial T}{\partial \dot{q}_2} + \ldots + \dot{q}_n\frac{\partial T}{\partial \dot{q}_n} = 2T$.

   (B) How the Non-holonomic constraints are different from holonomic constraints? Derive Lagrange's EOM for holonomic constraints. [10+10=20]

7. (A) A particle of mass m is moving in a plane under the action of a force $\mathbf{F}$. Using the generalized co-ordinates (r, θ), calculate the generalized forces for the particle.

   (B) Consider a double pendulum system in which a pendulum of mass m_2 is suspended from the pendulum of mass m_1, suspended from a support. The lengths of the inextensible strings of the two pendula are $\ell_1 = \ell_2 = \ell$. The double pendulum is set into oscillation in a vertical plane. Obtain the Lagrangian and equations of motion for the double pendulum for $m_1 = m_2 = m$.

   [10+10=20]

8. (A) What are canonical transformations needed for the Hamilton's formalism? Distinguish between the four distinct types of generating functions of a canonical transformation.

   (B) Define a Poisson bracket of two functions and hence show that the transformation $Q = \sqrt{e^{-2q} - p^2}$, $P = \cos^{-1}(pe^q)$ is canonical. Find the generator of the transformation.

   [10+10=20]

9. (A) Given the canonical transformations $Q = (q^2 + p^2)/2$ and $P = -\tan^{-1}(q/p)$. Evaluate the Poisson bracket $[Q, P]$.

   (B) Discuss the solution of a Hamilton-Jacobi equation for the case when the Hamiltonian does not explicitly depend on time. [10+10=20]

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M.A./M.Sc. Part – II Supply 2020 & Annual – 2021

Subject: Mathematics **Paper: IV-VI (opt. ix) (Electromagnetic Theory)** **Time: 3 Hrs. Marks: 100**

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

SECTION-I

1. (a) Workout the relationship between conductors and condensers.
   (b) Calculate the magnitude of the given point charge so that the electric field 64 cm distant is 3.14 N/C.
2. (a) In the presence of conducting plates, find the potential at any point in the electric field.
   (b) An infinite line charge produces a field of $4.52\times10^4\ N/C$ at a distance of 1.96 m. Calculate the linear charge density.
3. (a) Discuss the Kirchhoff's Law.
   (b) Work out the Poisson equation for the vector potential $\overline{A}$.
4. (a) Workout the relationship between conductivity and resistance.
   (b) Calculate the electromotance induced in a loop by a pair of long parallel wires carrying a variable current.
5. (a) Work out the electromotance induced in a loop by a pair of long parallel wires carrying a variable current.
   (b) A point charge Q is located at the origin; the potential at (1, 0, 0) is 20V while the potential at (0, 2, 0) is 10V. Find Q?

SECTION-II

6. (a) Discuss the Maxwell's equations in free space and material media.
   (b) Find the magnetic field between the circular plates of a parallel-plate capacitor that is charging using the Ampere-Maxwell law. The plates have a radius of R. The bordering field should be ignored.
7. (a) Discuss the plane electromagnetic waves in homogeneous and isotropic media.
   (b) Discuss the Lienard-Wiechert potentials for a moving charge.
8. (a) Work out the ratios of the amplitudes of the incident, reflected and transmitted waves for the case when the incident wave is polarized with its $\overline{E}$ vector parallel to the plane of incident.
   (b) In a hollow cylindrical waveguide, explain the transverse electric waves.
9. (a) Prove that a plane electromagnetic wave in free space has only transverse components of electric and magnetic field vectors.
   (b) The earth receives about $1300\,watts/m^2$ radiant energy from the sun. Assuming the energy in the form of plane monochromatic wave, and also assuming normal incidence, compute magnitude of electric and magnetic fields vectors in the sun light.

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M.A./M.Sc. Part – II Supply 2020 & Annual – 2021

Subject: Mathematics | **Paper: IV-VI (opt. x) [Operations Research]** | **Time: 3 Hrs.** | **Marks: 100**

NOTE: Attempt any FIVE questions selecting at least TWO questions from each section.

SECTION 1

Q.1 A manufacturer of wooden articles produces chairs and tables which requires two types of input mainly, wood and labour. The manufacturer knows that three (3) units of wood and one (1) unit of labour are required for a table, while two (2) units of each are required for a chair. The profit is Rs. 20 for each table and Rs. 16 for each chair. The total available resources for the manufacturer are 150 units of wood and 75 units of labour. The manufacturer wants to maximize his profit.

(a) Formulate the problem as a Linear Programming model. (10)

(b) Solve the problem using graphical method. (10)

Q.2 **(a)** Explain the concept of degeneracy in simplex method. (8)

(b) Apply M-method to the following problem: (12)

$$\text{Maximize } z = 2x_1 + 5x_2$$

subject to

$$3x_1 + 2x_2 \geq 6$$
$$2x_1 + x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

Q.3 **(a)** Solve the following problem using two-phase method. (10)

Minimize $z = 4x_1 + x_2$

subject to

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

(b) Use dual simplex method to solve the following problem. (10)

Minimize $z = 3x_1 + 2x_2$

subject to

$$3x_1 + x_2 \geq 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

Q.4 **(a)** Write the steps of Hungarian method. (8)

(b) The assignment cost of assigning any one operator to any one machine is given in the following table. (12)

Machine \ Operator	1	2	3	4
A	1	4	6	3
B	9	7	10	9
C	4	5	11	7
D	8	7	8	5

Find the optimal assignment.

P.T.O

Q.5

(a) Write a note on north-west corner method. (8)

(b) Consider the transportation problem in Table 1. If a unit from a source is not shipped out (to any of the destinations), a storage cost in incurred at the rate of Rs.5, Rs.4 and Rs.3 for the sources 1, 2 and 3 respectively. Use Vogel's starting solution and determine all the iterations leading to the optimum shipping schedule. (12)

Table1

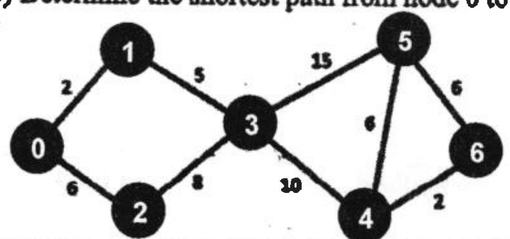
Rs.1	Rs.2	Rs.1	20
Rs.3	Rs.4	Rs.5	40
Rs.2	Rs.3	Rs.3	30
30	20	20	

SECTION 2

Q.6

(a) Write the steps of Floyd's algorithm. (8)

(b) Determine the shortest path from node 0 to node 6 using Dijkstra's method. (12)



Q.7

(a) Use the revised simplex method to solve the following problem. (10)

Maximize $z = 6x_1 - 2x_2 + 3x_3$

subject to

$$2x_1 - x_2 + 2x_3 \leq 2$$
$$x_1 + 4x_3 \leq 4$$
$$x_1, x_2, x_3 \geq 0$$

(b) Apply bounded-variables algorithm to solve the following problem involving bounded variables. (10)

Maximize $z = 3x_1 + 5x_2 + 2x_3$

Subject to

$$x_1 + 2x_2 + 2x_3 \leq 10$$
$$2x_1 + 4x_2 + 3x_3 \leq 15$$
$$0 \leq x_1 \leq 4, 0 \leq x_2 \leq 3, 0 \leq x_3 \leq 3$$

Q.8

Solve the following integer LP using branch and bound (B & B) method. (20)

Maximize $z = x_1 + x_2$

subject to

$$2x_1 + 5x_2 \leq 16$$
$$6x_1 + 5x_2 \leq 30$$

x_1, x_2 are non-negative integers.

Q.9

Use parametric linear programming to solve the following problem. (20)

Maximize $z = (3 - 6t)x_1 + (2 - 2t)x_2 + (5 + 5t)x_3$

Subject to

$$x_1 + 2x_2 + x_3 \leq 40$$
$$3x_1 + 2x_3 \leq 60$$
$$x_1 + 4x_2 \leq 30$$
$$x_1, x_2, x_3, t \geq 0$$

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Roll No. ....

M.A./M.Sc. Part – II Supply 2020 & Annual – 2021

Subject: Mathematics Paper: (IV-VI) (Opt. xii) (Advanced Functional Analysis) Time: 3 Hrs. Marks: 100

NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section.

SECTION – I

Q.1 (a) Define a complete metric space. Give at least three examples of incomplete metric supspaces of $\mathbf{R}^3$ under the usual metric.

(b) let $X = (X, \|\cdot\|)$ be normed space. Prove that there exists a Banach space $\widehat{X}$ and and Isometry A from X onto a subspace W of $\widehat{X}$ which is dense in $\widehat{X}$. Furthermore, $\widehat{X}$ is unique up to an isometry.

Q.2 (a) Prove that completeness is not a topological property. Prove that if X and Y are isometric metric spaces then if X is complete so is Y.

(b) prove that a discrete metric space on a nontrivial vector space can not be obtained from a norm.

Q.3 (a) Let N and M be normed space and $T : N \to M$ be a bounded linear operator. Then T is closed.

(b) State and Prove Parseval's Identity. How is it related with Totality of orthonormal sets in a Hilbert space.

Q.4 (a) Any two inner product spaces having the same finite dimension are isometrically isomorphic.

(b) State Bessel's inequality. Give its geometric interpretations and some applications.

SECTION – II

Q.5 (a) How can the separability of a normed space Y is related to the separability of the second Dual of Y. Can we find similar kind of relation between Y and its first dual.

(b) State and prove Banach Steinhass Theorem

Q.6 (a) If f be a sublinear functional on a vector space Y, then show that $M = \{x \mid f(x) \leq r, r > 0\}$ is a Convex Set.

(b) Give some applications of Principle of uniform boundedness.

Q.7 (a) Prove necessary and sufficient conditions for weak convergence of sequence in a normed space.

(b) Show that the projection operators defined as $T_j(x_1, x_2, ...x_n) = (x_j)$ are open.

Q.8 (a) Define adjoint operator and prove that it is linear and bounded.

(b) State and prove RiesZ's representation theorem for bounded linear functionals on $C[a, b]$.

Q.9 (a) a)Define adjoint operator of a Linear operator T and prove that it has the same norm as that of T.

(b) Let Y be subspace of $X = C[0, 1]$ which consists if all functions $f \in X$ which have a continuous derivative. Then show that the differential operator

$$T : Y \to X$$

defined as $T(f) = f'$, is closed.

UNIVERSITY OF THE PUNJAB

M.A./M.Sc. Part – II Supply 2020 & Annual – 2021 (CLASH)

Roll No. ....

Subject: Mathematics | Paper: IV-VI (opt. ix) (Electromagnetic Theory) | Time: 3 Hrs. | Marks: 100

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

SECTION-I

1. (a) Prove that the incident, reflected and transmitted waves are coplanar
(b) Two small identical conducting spheres have charges $2 \times 10^{-9}C$ and $-0.5 \times 10^{-9}C$, respectively, when they are placed 4cm apart. How much force they exert on each other?
2. (a) In the presence of conducting plates, find the potential at any point in the electric field.
(b) Along the z axis, two-point charges are separated by distance 'a'. When the charges are of opposing polarity but equal magnitude, find the electric field at any point in the z=0 plane.
3. (a) Show that there are effective charge distributions outside due to the polarized dielectric.
(b) Show that the free charge density decreases exponentially with time.
4. (a) Calculate the self-inductance of a toroid.
(b) The inductance of a closely wound N turns coil is such that an emf of 3mV is induced when the current changes at the rate of 5 A/sec. The steady current of 8 A produces a magnetic flux of 40 μ ωb through the turn. Calculate the inductance of the coil. How many turns does the coil have?
5. (a) Find the potential difference between two points "a" and "b" $(b > a)$ lying on spherical radial line $(\theta, \emptyset = constant)$ from the origin.

(b) In electric field given by

$$E = \frac{1.5}{\epsilon_0}x^2y^2\hat{\imath} + \frac{1}{\epsilon_0}x^3y\hat{\jmath}$$

how much charge lie within a cube, 4cm of side, if its geometric center is at origin and its sides are parallel to the coordinate axis.

SECTION-II

6. (a) For field vectors, drive the electromagnetic wave equations.
(b) Prove that the electric and magnetic energy densities for a plane electromagnetic wave in free space are equal.
7. (a) Demonstrate that the Poynting vector extends radially throughout the cylindrical volume.
(b) Discuss the propagation of plane electromagnetic waves in non-conductors.
8. (a) Work out the coefficients of reflection and transmission at an interface using Fresnel's equations for case when $\overline{E}$ is polarized normal to the plane of incidence.
(b) In a hollow cylindrical waveguide, explain the transverse electric waves.
9. (a) State and prove Snell's law
(b) If $f_1(x,t) = A_1cos(kx - \omega t + \theta_1)$ and $f_2 = A_2cos(kx - \omega t + \theta_2)$ are two sinusoidal waves then show that their sum is also a sinusoidal wave.

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M.A./M.Sc. Part – II Supply 2020 & Annual – 2021 (CLASH)

Roll No. ....

Subject: Mathematics | Paper: IV-VI (opt.vii) [Quantum Mechanics] | Time: 3 Hrs. | Marks: 100

NOTE: Attempt any FIVE questions by selecting atleast TWO from each section.

Section-I

1. (a) State Heisenberg uncertainty principle and obtain uncertainty Δx for the wavefunction given by (10)

$$\psi(x,t) = \sqrt{\frac{1}{a\sqrt{2\pi}}}\exp\left[-\frac{(x-x_0)^2}{4a^2}\right]\exp\left(\frac{ip_0x}{\hbar}\right)\exp\left(-i\omega_0 t\right)$$

(b) The work function of zinc is 3.6 eV. What is the energy of the most energetic photo-electron emitted by ultraviolet light wavelength $\lambda = 2500$ Å ($\AA = 10^{-10}$m and $h = 4.1357 \times 10^{-15}$eV/sec). (10)

2. (a) Show that $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + \left[\hat{A},\hat{B}\right] + \frac{1}{2!}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right] + \frac{1}{3!}\left[\hat{A},\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]\right] + \ldots$ (10)

(b) Measurement of the position of the particle in a 1-D box with walls at $x = 0$ and $x = a$ finds the value $x = a/2$. Show that in the subsequent measurement, (10)

- it is equally probable to find the particle in any odd-energy eigenstate ($n = 1, 3, \ldots$).
- the probability of finding particle in any even-energy is zero($n = 2, 4, \ldots$).

3. (a) Let $\hat{\mathbf{P}}_\pm = \frac{1\pm\hat{\mathbf{P}}}{2}$ be the projections of parity operator $\hat{\mathbf{P}}$, defined by $\hat{\mathbf{P}}_\pm f(x) = f_\pm$, where (10)

$$f_\pm = \frac{f(x) \pm f(-x)}{2}.$$

Show that

- $(\hat{\mathbf{P}}_\pm)^2 = \hat{\mathbf{P}}_\pm$
- $[\hat{\mathbf{P}}_+, \hat{\mathbf{P}}_-] = 0$

(b) Show that eigenvalue of Hermitian operator are always real and the corresponding eigenfunctions are orthogonal. (10)

4. (a) Derive reflection and transmission coefficient for rectangular barrier scattering where the energy E of the particles in the incident beam is greater than the height V of potential barrier. (10)

(b) Given that $\phi_n = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$ are the normalized eigenstates of particle in 1-D box problem with eigenenergies $E_n = n^2E_1$, $n = 1, 2, 3, \ldots$.
If initial state of the particle is given by the superposition $\psi(x,0) = \frac{2}{\sqrt{5}}\phi_1 + \frac{1}{\sqrt{5}}\phi_2$
Find the evolved wave function $\psi(x,t)$. Further show that expectation value of the energy operator $\hat{H}$ remains constant.

P.T.O

Section-II

5. (a) Let $|n\rangle$ be the n^{th} eigenfunction (Dirac notation) of quantum mechanical harmonic oscillator. If $\hat{a}$ and $\hat{a}^\dagger$ denote annihilation and creation operators, respectively, then evaluate (10)
   - $\langle n|\hat{x}|n\rangle$,
   - $\langle n|\hat{x}^2|n\rangle$; where $\hat{x} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right)$.

   (b) Find the first order corrections, $E_n^{(1)}$, to the eigen energies of Anharmonic oscillator whose perturbation Hamiltonian is given by $\hat{H}' = K'x^4$, where $K' \ll 1$ and $\hat{x} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right)$. (10)

6. (a) How does the twofold-degenerate energy $E = 2\hbar\omega_0$ of the two-dimensional harmonic oscillator separate due to the perturbation $H' = K'xy$, where $\hat{x} = \frac{1}{\sqrt{2\beta}}\left(\hat{a} + \hat{a}^\dagger\right)$ and $\hat{y} = \frac{1}{\sqrt{2\beta}}\left(\hat{b} + \hat{b}^\dagger\right)$?

   (b) Find expressions for first order corrections to the eigenvalues and eigenfunctions for very small perturbation to the time independent, non-degenerate Hamiltonian $\hat{H}_0$. (10)

7. (a) Suppose that a rigid rotator is in eigenstate of $\hat{L}^2$ and $\hat{L}_z$ corresponding to $l = 1$ and $m = +1$. What is the probability that measurement of L_x finds the respective values $m = 0, \pm 1$? (10)

   (b) Prove the following statements. (10)
   - $[\hat{L}_+, \hat{L}_-] = 2\hbar\hat{L}_z$, where $\hat{L}_\pm$ are ladder operators.
   - If ϕ_m is an eigenfunction of $\hat{L}_z$ for eigenvalue $\hbar m$, then show that $\hat{L}_+\phi_m$ is also an eigenfunction of $\hat{L}_z$ corresponding to eigenvalue $\hbar(m+1)$.

8. (a) Explain scattering cross section for 3-D scattering and obtain general expression. (10)

   (b) Using Born's approximation, compute the phase shift δ_1 for scattering in a centrally symmetric field. (10)

9. (a) Obtain the radial part of the Scrhödinger's wave equation. Also find the allowed eigen energies for infinite spherical well given by (10)

$$V(r) = \begin{cases} 0, & r \le a; \\ \infty, & r > a. \end{cases}$$

   (b) Compute the expression for ionization potential of hydrogen, helium and lithium atoms. (10)