# UNIVERSITY OF THE PUNJAB

Roll No. ...............

**M.A./M.Sc. Part – I Annual Examination – 2020**

Time: 3 Hrs. Marks: 100

Subject: Mathematics (Old & New Course) Paper: I (Real Analysis)

***NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section.***

**SECTION – I**

1. (a) Let $x, y, a$ and $b$ are positive real numbers such that $\frac{x}{y} < \frac{a}{b}$. Show that (10 marks)

$$\frac{x}{y} < \frac{x+a}{y+b} < \frac{a}{b}.$$

(b) If $X = (x_n)$ is a sequence of real numbers then prove that there is a subsequence of $X$ that is monotone. (10 marks)

2. (a) Let $(x_n : n \in \mathbb{N})$ be a sequence such that there exist $A > 0$ and $C \in (0, 1)$ for which $|x_{n+1} - x_n| \leq AC^n$ for any $n \geq 1$. Show that $(x_n)$ is Cauchy. Is this conclusion still valid if we assume only $Lim_{n\to\infty}|x_{n+1} - x_n| = 0$. (10 marks)

(b) Suppose $S$ is a non-empty set of real numbers which is bounded above and let $\alpha \in \mathbb{R}$, then prove that (10 marks)

$$Sup(\alpha + S) = \alpha + Sup(S).$$

3. (a) Apply the definition of limit to evaluate: (10 marks)

$$Lim_{x\to 3}\frac{2x+3}{4x-9} = 3.$$

Also find the value of the corresponding $\delta$.

(b) Show that every continuous function on a closed and bounded interval $I \subset \mathbb{R}$ is uniformly continuous. (10 marks)

4. (a) Let $A,\ B \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions such that $f(A) \subseteq B$. If $f$ is continuous at a point $c \in A$ and $g$ is continuous at $b = f(c) \in B$, then the composition function $gof : A \to \mathbb{R}$ is continuous at $c$. Let $g_3(x) = \sin x$ for $x \in \mathbb{R}$ is continuous on $\mathbb{R}$ and let $f : A \to \mathbb{R}$ is continuous on $A$. Show that $g_3 of : A \to \mathbb{R}$ is continuous on $A$. (10 marks)

(b) Show that the limit

$$Lim_{x\to 0}\frac{\sin x}{\sqrt{1-\cos x}}$$

does not exist in $\mathbb{R}$. (10 marks)

5. (a) Let $f : [a, b] \to \mathbb{R}$ be a function continuous on $[a, b]$ and differentiable on $(a, b)$. Show that there exists at least one real number $a < c < b$ such that (10 marks)

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

(b) Establish the following inequality: (10 marks)

$$\frac{x}{1+x} < ln(1+x) < x,\ \forall x > 0$$

P.T.O.

## SECTION – II

6. (a) Let $f \in \mathcal{R}(\alpha)$ on $I := [a, b]$ and if $a < c < b$, then $f \in \mathcal{R}(\alpha)$ on $I_1 := [a, c]$ and $I_2 := [c, b]$ and (10 marks)

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha.$$

(b) Evaluate (10 marks)

$$\int_0^\pi (x+1)\, d(\sin x + \cos x).$$

7. (a) Prove that

$$f(x) = \begin{cases} x \cos\left(\frac{\pi}{2x}\right), & 0 < x \leq 1, \\ 0, & x = 0, \end{cases}$$

is not of bounded variation on $[0, 1]$. (10 marks)

(b) Consider the sequence $\{f_n\}$ defined by $f_n(x) = \frac{nx}{e^{nx}}$, for $x \in [0, 2]$. Show that $lim_{n\to\infty} f_n(x) = 0$ for $x \in (0, 2]$. Show that the convergence is not uniform on $[0, 2]$. (10 marks)

8. (a) Give examples to illustrate that (i) the pointwise limit of continuous (respectively, differentiable) functions is not necessarily continuous (respectively, differentiable), (ii) the pointwise limit of integrable functions is not necessarily integrable. (10 marks)

(b) Let $f$ be a real valued continuous function defined on $[a, b]$. If $f'$ exists and is bounded on $[a, b]$, then $f$ is a function of bounded variations on $[a, b]$. But its converse may not be true. (10 marks)

9. (a) For $\varepsilon > 0$, there exists a partition such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon,$$

then prove that the following statements are equivalent. (10 marks)

(i) If expression (1) holds for partition $P = \{x_0, x_1, x_2, ..., x_n\}$ and $s_i$, $t_i$ are arbitrary points in $[x_{i-1}, x_1]$, then

$$\sum_{i=1}^{n} = |f(s_i) - f(t_i)|\Delta\alpha_i < \varepsilon.$$

(ii) If $f \in \mathcal{R}(\alpha)$ and hypothesis (1) holds, then

$$|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_x^b f d\alpha| < \varepsilon.$$

(b) Test the convergence of (10 marks)

$$\int_0^\infty \frac{dx}{1 + x^4 \cos^2 x}$$

# UNIVERSITY OF THE PUNJAB

Roll No. ...............

**M.A./M.Sc. Part – I Annual Examination – 2020**

**Subject: Mathematics (Old & New Course)** **Paper: II (Algebra)** **Time: 3 Hrs.** **Marks: 100**

***NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section.***

## SECTION – I

Q.1. (a) Prove that derived subgroup of a group $G$ is a normal subgroup of $G$. (10)

(b) Find normal subgroups of the alternating group $A_4$. (10)

Q.2. (a) Let $\Phi: G_1 \rightarrow G_2$ be a group homomorphism. Prove that Ker $\Phi = \{e\}$ if and only if $\Phi$ is injective. (10)

(b) Find the conjugacy classes of the Dihedral group (10)

$$D_3 = (a, b : a^3 = e = b^2, bab^{-1} = a^{-1}).$$

Q.3. (a) Prove that Aut$(V_4) \cong S_3$, where $V_4 = [a, b : a^2 = b^2 = e = (ab)^2]$ and $S_3$ is the symmetric group. (10)

(b) Show that a homomorphic image of a cyclic group is cyclic. (10)

Q.4. (a) Define the characteristic subgroup of a group $G$. Prove that center of $G$ is a characteristic subgroup of $G$. (10)

(b) Let $H = (a : a^3 = e)$ and $K = (b : b^2 = e)$ be two multiplicative cyclic groups. Show that $H \times K$ is a cyclic group. (10)

Q.5. (a) Prove that any two Sylow $p$-subgroups of a group $G$ are conjugate to each other. (10)

(b) Find Sylow 2-subgroups and Sylow 3-subgroups of the symmetric group $S_3$. (10)

## SECTION – II

Q.6. (a) Define a Field. Show that $(Z_7, +, \cdot)$ is a Field. (10)

(b) Prove that a finite integral domain forms a field. (10)

Q.7. (a) If $J_1$ and $J_2$ are any two ideals of a ring $R$. Show that $J_1 \cap J_2$ is an ideal of $R$. (10)

(b) Let $R$ be a ring and $\Phi : R \rightarrow R/\text{Ker}\,\Phi$ be the natural map. Prove that $\left(\frac{R}{\text{Ker}\phi}, +, \cdot\right)$ is a ring structure. (10)

Q.8. (a) Let $\Phi : R \rightarrow S$ be a ring homomorphism. If $\Phi$ is an epimorphism, then show that

$$R/\text{Ker}\,\Phi \cong S.$$ (10)

(b) Let $V$ be a finite dimensional vector space over a Field $F$ and $T : V \rightarrow V$ be a homomorphism. Then prove the Rank-Nullity therorem: (10)

$$\text{Dim}(V) = \text{Rank}(T) + \text{Nullity}(T).$$

Q.9. (a) Find a real orthogonal matrix $P$, if possible, for which $P^{-1}AP$ is diagonal, where (10)

$$A = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}.$$

(b) Let $U(F)$ and $W(F)$ be two vector spaces. Then the set $Hom(U,W)$ of all homomorphism forms a vector space over the same field $F$. (10)

# UNIVERSITY OF THE PUNJAB

Roll No. ................

M.A./M.Sc. Part – I Annual Examination – 2020

Subject: Mathematics (Old & New Course)
Paper: III (Complex Analysis and Differential Geometry)

Time: 3 Hrs. Marks: 100

***NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section. All Questions carry equal marks.***

## SECTION – I

**Q 1.** (a) Prove that $f(z) = z^2$ is continuous in the region $|z| \leq 1$.
(b) Show that $f(z) = |z|^2$ is differentiable only at $z = 0$.

**Q 2.** (a) Derive the Laplace equation in polar form.
(b) If $u(x, y) = e^{-x}(x \sin y - y \cos y)$, then find $v$ such that $f(z) = u + iv$, is analytic.

**Q 3.** (a) Define linear fractional transformation. Find the linear fractional transformation, which maps the points $z_1 = -1, z_2 = 0, z_3 = 1$ onto the points $w_1 = -i, w_2 = 1, w_3 = i$ respectively.
(b) State only Mittag-Leffler's Expansion Theorem, Weierstrass Factorization Theorem.

**Q 4.** (a) Find all the roots of the equation $\sin z = \cos h4$.
(b) Find all the value of the integral $\int_c f(z)dz$ if $f(z) = e^z$ and $C$ is the arc from $z = \pi$ to $z = 1$, consisting of: (i) The line segment joining these points. (ii) The portion of the coordinate axes joining these points.

**Q 5.** (a) Evaluate $\oint_c \frac{dz}{z-a}$; where $c$ is any simple closed curve and (i) $z = a$ is outside $c$ (ii) $z = a$ is inside $c$.
(b) Find the residue of $f(z) = cosec^2 z$ at $z = 0$

## SECTION – II

**Q 6.** (a) If the parametric curves are orthogonal. Show that the differential equation of the lines on the surface cutting the curve $u = constant$ at constant angle $\beta$ is

$$\frac{du}{dv} = \tan \beta \sqrt{\frac{G}{E}}$$

(b) Find the principle curvature and the line of curvature on the surface $x = u \cos \phi$, $y = u \sin \phi$, $z = c\phi$.

**Q 7.** (a) Taking $x, y$ as parameters calculate the fundamental magnitude and the normal of the surface $2z = ax^2 + 2hxy + by^2$.
(b) Prove that $(i)$ $H\vec{n} \times n_1 = M\vec{r_1} - L\vec{r_2}$ $(ii)$ $H\vec{n} \times n_2 = N\vec{r_1} - M\vec{r_2}$

**Q 8.** Prove that the necessary and sufficient condition for parametric curves to the line of curvature is $F = 0$ and $M = 0$.
(b) Define: Curve, Surface, Tangent, Curvature, Torsion.

**Q 9.** Show that the sum of normal curvatures in two directions at right angle is equal to the sum of principal curvatures.
(b) Spheres of constant radius $b$ have their centers on the fixed circle $x^2 + y^2 = a^2$, $z = 0$. Prove that their envelope is the sphere.

# UNIVERSITY OF THE PUNJAB

**M.A./M.Sc. Part – I Annual Examination – 2020**

Roll No. ................

**Subject: Mathematics (Old & New Course) Paper: IV (Mechanics)** **Time: 3 Hrs. Marks: 100**

***NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section. All questions carry equal marks.***

## SECTION – I

1. (a) If $\phi(x,y,z) = xy^2z$ and $\mathbf{A} = xz\,\mathbf{i} - xy^2\,\mathbf{j} + yz^2\,\mathbf{k}$, find $\frac{\partial^3}{\partial x^2 \partial z}(\phi \mathbf{A})$ at the point $(2,-1,1)$.

   (b) The quantities ii, ij, ik, ji, jj, jk, ki, kj, kk (no dot or cross product) are called unit dyads. A dyadic is the sum of dyads with suitable coefficients. Give a possible definition of $(\mathbf{A} \times \nabla)\mathbf{B}$ at the point $(1,-1,1)$ for $\mathbf{A} = xz\,\mathbf{i} - y^2\,\mathbf{j} - yz^2\,\mathbf{k}$ and $\mathbf{B} = 2z^2\,\mathbf{i} - xy\,\mathbf{j} + y^3\,\mathbf{k}$. [10+10=20]

2. (a) What is divergence theorem? Verify the divergence theorem for $\mathbf{A} = 4x\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k}$ taken over the region bounded by $x^2 + y^2 = 4$ and $z = 3$.

   (b) Evaluate the integral $\iiint_V \mathbf{F}\, dV$ for $\mathbf{F} = 2xz\mathbf{i} - x\mathbf{j} + y^2\mathbf{k}$ where $V$ is the region bounded by the surfaces $x = 0, y = 0, y = 6, z = x^2, z = 4$. [10+10=20]

3. (a) Verify that if $F = (2xy + z^3)\,\mathbf{i} + x^2\,\mathbf{j} + 3xz^2\,\mathbf{k}$ is a conservative force field or not. If so, then find the scalar potential and the work done in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$. [10+10=20]

   (b) The coordinate transformation between the rectangular coordinates $(x,y,z)$ and the elliptic cylindrical coordinates $(u,v,z)$ is given by $x = a\cosh u \cos v, y = a \sinh u \sin v, z = z$. Find the expression for the unit vectors in the elliptic cylindrical coordinates and their time derivatives

4. (a) The contravariant components of a tensor in rectangular coordinates are $yz, 3, 2x+y$. Find its covariant components in parabolic cylindrical coordinates.

   (b) Given that $x = uv\cos\phi,\ y = uv\sin\phi,\ z = \frac{1}{2}(u^2 - v^2)$, the coordinate transformation between the rectangular coordinates $(x,y,z)$ and the paraboloidal coordinates $(u,v,\phi)$. Determine the Christoffel symbols of two kinds for this coordinate system. [10+10=20]

5. (a) Define the terms: curvilinear coordinates, orthogonal curvilinear coordinates, coordinate curves and coordinate surfaces, covariant bases vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and contravariant bases vectors $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$. Give the explicit relationship between them.

   (b) Let $(x^j)$ and $(\bar{x}^j)$ be two general curvilinear coordinate systems. Prove that $\frac{\partial^2 x^m}{\partial \bar{x}^j \partial \bar{x}^k} = \bar{\Gamma}^n_{jk} \frac{\partial x^m}{\partial \bar{x}^n} - \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\Gamma^m_{pq}$, where $\Gamma^m_{pq}$ are the Christofell symbols of second kind. [10+10=20]

## SECTION II

6. (a) For any time dependent vector function $\mathbf{A}(t)$, obtain a relationship between fixed and rotating coordinate systems. Prove that the angular acceleration is the same in the two coordinate systems.

   (b) The moments and products of inertia of a rigid body about the x, y and z axes are $I_{xx} = 3, I_{yy} = 10/3, I_{zz} = 8/3,\ I_{xy} = 4/3,\ I_{xz} = -4/3,\ I_{yz} = 0$. Find the principal moments of inertia and the directions of principal axes. [10+10=20]

7. (a) State and prove parallel axes theorem for the moments and products of inertia for a continuous distribution of mass.

   (b) A rigid body consists of 3 particles of masses 2,1,4 located at $(1,-1,1), (2,0,2), (-1,1,0)$ respectively. Find the angular momentum of the body if it is rotated about the origin with angular velocity $\omega = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$. [10+10=20]

8. (a) Find the moment of inertia of uniform circular cone of mass $M$, height $h$ and circular base of radius $a$ about (i) its axis of symmetry and (ii) the diameter of its base.

   (b) Derive Euler's equations of motion of a rigid body rotating relative to a set of coordinates axes coinciding with the principal axes of inertia fixed in the body and establish the constancy of kinetic energy and angular momentum. Write Eulers equations of motion in case the axes are not principal axes. [10+10=20]

9. (a) What are Euler angles? Express the components of angular velocity in terms of Euler angles.

   (b) What is a spinning top? Derive the equations of motion of the spinning top in terms of Euler angles, when one of the points on its axis is fixed. Deduce the condition for a sleeping top. [10+10=20]

# UNIVERSITY OF THE PUNJAB

**M.A./M.Sc. Part – I Annual Examination – 2020**

**Roll No. ...............**

**Subject: Mathematics (Old & New Course)**
**Paper: V (Topology and Functional Analysis)**

**Time: 3 Hrs. Marks: 100**

***NOTE: Attempt any FIVE questions in all selecting at least TWO questions from each section.***

## SECTION – I

**Q.1 (a)** Let $X=\{1,2,3,4,5\},\Im=\{\phi,\{1\},\{3,4\},\{1,3,4\},\{2,3,4,5\},X\}$. Find the interior, closure, exterior and frontier of the set $A=\{2,3,4\}$. (10)

**(b)** Let X and Y be topological spaces. Prove that a function $f:X\to Y$ is continuous on X if and only if for each subset $V$ closed in $Y$, $f^{-1}(V)$ is closed in X. (10)

**Q.2 (a)** (i) Let $\Omega$ be a non-empty collection of subsets of $X$ such that $X=\cup\Omega$. Then prove that $\Omega$ is a sub-base for some topology on $\Omega$. (10)

(ii) Let $X=\{a,b,c,d\},\Omega=\{\{a\},\{b,c\},\{b,d\}\}$ find the topology generated by $\Omega$.

**(b)** A Topological space $(X,\Im)$ is normal if and only if for any closed set $A$ and an open set $U$ containing $A$ there is at least one open set $V$ containing $A$ such that $A\subseteq V\subseteq\bar{V}\subseteq U$. (10)

**Q.3 (a)** Prove that closed subspaces of a Lindelof space are Lindelof. (10)

**(b)** Prove that every compact hausdorff space is normal. (10)

**Q.4 (a)** Prove that every sequentially compact space is countably compact space. (10)

**(b)** Let $\{A,B\}$ be disconnection of a space $X$ and $C$ be a connected sub-space of $X$. Then prove that $C$ is contained in either $A$ or $B$. (10)

## SECTION – II

**Q.5 (a)** Show that $d(x,y) = \sqrt{|x-y|}$ defines a metric on the set of all real numbers. (10)

**(b)** Prove that the space $C$ of complex numbers with usual metric is complete. (10)

**Q.6 (a)** (i) Let $f:X\to Y, g:Y\to Z$ be uniformly continuous. Then prove that $g\circ f:X\to Z$ is uniformly continuous. (10)

(ii) Let $A$ be a non-empty subset of $(X,d)$. Then the function $f:X\to\mathbb{R}$ defined by $f(x)=d(x,A)$ is uniformly continuous.

**(b)** Define equivalent norms. Also show that the following norms on $R^n$ are equivalent (10)

$$\|x\|_0 = \sup_{i=1}^{n}|x_i|,\ \|x\|_1 = \sum_{i=1}^{n}|x_i|,\ \|x\|_2 = \sqrt{\sum_{i=1}^{n}|x_i|^2}\ \ \forall x=(x_1,x_2,...x_n)\in R^n.$$

**Q.7 (a)** (i) Define convex set. For any convex set $C$ in a linear space $N$ and for any scalars $\alpha\ge 0,\beta\ge 0$, show that $(\alpha+\beta)C=\alpha C+\beta C$. (10)

(ii) For any $x=(x_1,x_2,...,x_n)\in R^n$ define $f:\mathbb{R}^n\to\mathbb{R}$ by $f(x)=\sum_{i=1}^{n}x_i$ then prove that $f$ is continuous linear functional. Also find its norm.

**(b)** Let $N$ be a normed space in which every closed and bounded subset is compact. Then prove that $N$ is a Banach space. (10)

**Q.8 (a)** If $M$ is a Banach space then prove that $B(N,M)$ is Banach space under the norm given by $\|T\| = \sup_{\|x\|=1}\|Tx\|, x\in N, T\in B(N,M)$. (10)

**(b)** Let $N$ be an n-dimensional normed space then prove that its dual $N^*$ also is n-dimensional. (10)

**Q.9 (a)** Let $\{x_n\},\{y_n\}$ be any sequences in an inner product space V. (10)

(i) If $x_n\to x, y_n\to y$ then show that $\langle x_n,y_n\rangle\to\langle x,y\rangle$

(ii) If $\{x_n\},\{y_n\}$ are Cauchy sequences in V, then show that $\langle x_n,y_n\rangle$ is convergent sequence in F where F is R or C.

**(b)** Prove that every Hilbert space is reflexive. (10)